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Chebyshev Subspaces in the Space of Compact Operators

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INTRODUCTION

Noncommutative approximation and, in particular, approximation in spaces of operators has received considerable attention recently, as evidenced in [4–6, 8] and other articles. The purpose of this paper is to characterize the semi-Chebyshev subspaces of $\mathscr{C}(\mathscr{H})$, the algebra of compact operators on an infinite-dimensional Hilbert space. The underlying Hilbert space may be either real or complex. However, the notation corresponding to a complex Hilbert space will be used throughout the paper. Our conditions are reminiscent of those used in classifying the semi-Chebyshev subspaces of $C[a, b]$ and c_0 and lead to the result that there exist Chebyshev subspaces of every finite dimension in $\mathscr{C}(\mathscr{H})$, $\mathscr{H}$ separable. An intrinsic characterization of the finite-dimensional Chebyshev subspaces is then obtained. However, unlike $C[a, b]$ and c_0, there does not seem to be any satisfactory concept of an interpolation subspace in $\mathscr{C}(\mathscr{H})$. In Section 3, the finite-codimensional proximinal subspaces of $\mathscr{C}(\mathscr{H})$ are characterized, leading to the interesting result that, just as in the case of c_0, there are no finite-codimensional Chebyshev subspaces in $\mathscr{C}(\mathscr{H})$. The final result indicates that all closed finite-codimensional subspaces of $\mathscr{C}(\mathscr{H})$ are very strongly non-Chebyshev.

DEFINITIONS AND NOTATION. Throughout this paper we will use the representation of the dual space of $\mathscr{C}(\mathscr{H})$ as τ .c., the space of trace class operators. From [9] we note the following pertinent results: any $C \in \mathscr{C}(\mathscr{H})$ may be expressed in its Schmidt expansion as $C = \sum \lambda_n v_n \otimes \bar{u}_n$ with the sum over a countable index set, $\lambda_n > 0$ for all n, $\lambda_n \to 0$ (if the index set is infinite), and $\{u_n\}$, $\{v_n\}$ are orthonormal sets. $T \in \tau \cdot$ c. if $T \in \mathscr{C}(\mathscr{H})$, $T = \sum \lambda_n v_n \otimes \bar{u}_n$, and $\sum \lambda_n < \infty$; the norm in τ.c. is $|||T||| = \sum \lambda_n$ and if $T \in \tau$.c. and $C \in \mathscr{C}(\mathscr{H})$, $T(C) = \sum \langle TCu_n, u_n \rangle$, where $\{u_n\}$ is any orthonormal basis in $\mathscr{H}$. Let T^* denote the adjoint of T, $|T| = (T^* T)^{1/2}$, $\sigma(T) =$ spectrum of T, and $\pi_0(T) =$ the point spectrum of T. Let V be a closed subspace of a Banach space E and let $\mathscr{P}(T) = \{$set of best approximants to T from $V\}$. Then V is semi-Chebyshev, Chebyshev, or proximinal in E if cardinality $\mathscr{P}(T)$ is $\leqslant 1$, $=1$, $\geqslant 1$ for all $T \in E$. The subspace V is said to be factor reflexive if X/V is reflexive. Also, let $V^0 = \{T \in E: 0 \in \mathscr{P}(T)\}$, $S(V^\perp) = \{\phi \in E^* \mid \|\phi\| = 1, \phi(V) = 0\}$ and $U(E) =$ the closed unit ball of E.

1. EXTRINSIC CHARACTERIZATIONS

The following two theorems should be compared to the theorems concerning Chebyshev subspaces of $C(X)$ given in [2, 7]. Theorems 1 and 2 are based on generalizations of the concepts of α sets and θ sets given in [2]. For what follows we assume that V is a closed subspace of $\mathscr{C}(\mathscr{H})$.

DEFINITION 1. A *generalized* α *set* is the maximal pair of subspaces $(M, N) \subset \mathscr{H} \times \mathscr{H}$ such that X (resp. X^*) attains its norm on M (resp. N) for some $X \in S(V^0)$ (that M and N are subspaces readily follows from the Schmidt decomposition of X.)

DEFINITION 2. A *generalized* θ *set* is the pair of subspaces

$$(\mathscr{R}(T), \mathscr{R}(T^*)) \subset \mathscr{H} \times \mathscr{H}$$

for some $T \in S(V^\perp)$ that attains its norm on $S(\mathscr{C}(\mathscr{H}))$.

We say that C is zero on a generalized α set if $C(M) = 0 = C^*(N)$. (C being 0 on a generalized θ set has the analogous meaning.)

THEOREM 1. *$V \subset \mathscr{C}(\mathscr{H})$ is semi-Chebyshev if and only if 0 is the only element of V that vanishes on a generalized α-set.*

For the proof we need the following lemma:

LEMMA 1. *Let $M \subset \mathscr{H}$ be the subspace on which $X \in \mathscr{C}(\mathscr{H})$ attains its norm and suppose $X(M) = N$; then $X(M^\perp) \subset N^\perp$, where $M^\perp$ is the orthogonal complement of M.*

Proof of lemma. Write X in its Schmidt expansion, i.e., $X = \sum \lambda_n v_n \otimes \bar{u}_n$. Then $M = \operatorname{span}\{u_n \mid \lambda_n = \lambda_1\}$ and the result is clear by the form of the expansion.

Proof of theorem. ($\Leftarrow$) This implication can be obtained by the same argument as in [2], applied both to the operators and to their adjoints.

($\Rightarrow$) Suppose $0 \neq C \in V$ is zero on the generalized α set associated with X. Thus $\mathscr{R}(C) \subset (\ker C^*)^\perp \subset N^\perp$, $M \subset \ker C$, and $\| X|_{M^\perp} \| = \| X \| - \epsilon$ for some $\epsilon > 0$. We may assume that $\| C \| \leqslant \epsilon$. By Lemma 1, for any $u \in \mathscr{H}$, we may write $u = u_1 + u_2$, $u_1 \in M$, and $u_2 \in M^\perp$ and we have:

$$\|(X - C)\, u\|^2 = \|(X - C)\, u_1 + (X - C)\, u_2\|^2 = \| Xu_1 \|^2 + \|(X - C)\, u_2\|^2.$$

Thus, $\| X - C \| = \max\{\| X|_M \|, \|(X - C)_{M^\perp}\|\} = \| X \|$. Hence, $C \neq 0$ is also a best approximant to X. Q.E.D.

Before proving our next theorem we need the following:

LEMMA 2. *$T \in \tau$.c. attains its norm on $X \in S(\mathscr{C}(\mathscr{H}))$ if and only if corresponding to the Schmidt decomposition of $T = \sum_1^N \lambda_n v_n \otimes \bar{u}_n$, we have $X(v_n) = u_n$, $n = 1,\ldots, N$.*

Proof. ($\Leftarrow$) This is a simple computation.

($\Rightarrow$) Because the trace class operators that attain their norm are exactly the finite-rank operators, we have the Schmidt decomposition

$$T = \sum_1^N \lambda_n v_n \otimes \bar{u}_n .$$

Therefore,

$$|||\, T \,||| = \sum_1^N \lambda_n = T(X) = \sum_1^N \lambda_n \langle X v_n, u_n \rangle \leqslant \sum_1^N \lambda_n .$$

It is evident that the above inequality is an equality only if $X(v_n) = u_n$, $n = 1,\ldots, N$. Q.E.D.

THEOREM 2. *A subspace $V \subset \mathscr{C}(\mathscr{H})$ is semi-Chebyshev if and only if 0 is the only element of V that vanishes on a generalized θ set.*

Proof. Suppose there exists a nonzero $C \in V$ such that C vanishes on a generalized θ set corresponding to some T. Since T attains its norm, the Schmidt expansion of T has the form

$$T = \sum_1^N \lambda_n v_n \otimes \bar{u}_n .$$

Consider the operator $X = \sum_1^N u_n \otimes \bar{v}_n$. Since $T(X) = 1 = \| X \|$, by [10, p. 18] we have that $X \in V^0$. Now note that the α set associated with X is $(\text{span}\{v_n\}_1^N, \text{span}\{u_n\}_1^N) = (\mathscr{R}(T), \mathscr{R}(T^*))$. Since C vanishes on the generalized θ set associated with T, C vanishes on the generalized α set associated with X; hence, by Theorem 1, V is not semi-Chebyshev.

To prove the converse, suppose V is not semi-Chebyshev. Then by [10, p. 105], there exists $T \in S(V^\perp)$, $X \in S(V^0)$, and $0 \neq C \in V$ such that $T(X) = \| X \| = \| X - C \| = 1$. By Lemma 2, if T has the form $T = \sum_1^N \lambda_n v_n \otimes u_n$, we have $X(v_n) = u_n$ and $(X - C)(v_n) = u_n$. Hence, $C(v_n) = 0$ $n = 1,..., N$, i.e., $C(\mathscr{R}(T)) = 0$. Since

$$T^*(X^*) = \| X^* \| = \| X^* - C^* \| = 1,$$

we have $C^*(\mathscr{R}(T^*)) = 0$, and thus, C vanishes on the generalized θ set associated with T. Q.E.D.

It is interesting to note that every generalized θ set is a generalized α set and every generalized α set contains a generalized θ set. The first statement follows from the proof of the first implication in Theorem 2. For the second statement, let (M, N) be a generalized α set corresponding to X. Thus, there exists a $T \in S(V^\perp)$ so that $T(X) = \| X \| = 1$. If $T = \sum_1^N \lambda_n v_n \otimes \bar{u}_n$, then $X(v_n) = u_n$ for all n by Lemma 2. Thus, X attains its norm on $\mathscr{R}(T)$ implying $\mathscr{R}(T) \subset M$. Similarly $\mathscr{R}(T^*) \subset N$, proving the statement.

As an application of our previous theorems, we have:

THEOREM 3. *Let $\mathscr{H}$ be a separable Hilbert space. Then $\mathscr{C}(\mathscr{H})$ has N-dimensional Chebyshev subspaces for each positive integer N.*

Proof. Let N be fixed and let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis in $\mathscr{H}$. Define $C_j = \sum_{n=1}^\infty (1/n)\, e_{Nn+j} \otimes \bar{e}_n$, $j = 1,..., N$. Evidently, C_j is a compact operator for all j and $0 \notin \pi_0(C_j)$, $j = 1,..., N$. Let $V_N = \text{span}\{C_j\}_{j=1}^N$. Since $\mathscr{R}(C_j) \perp \mathscr{R}(C_i)$ for $i \neq j$, no linear combination of the C_j has 0 in its point spectrum. By Theorem 1, V_N is Chebyshev.

2. INTRINSIC CHARACTERIZATIONS

There is a very simple intrinsic characterization of the one-dimensional Chebyshev subspaces of $\mathscr{C}(\mathscr{H})$.

THEOREM 4. *Given a nonzero* $C \in \mathscr{C}(\mathscr{H})$, $V =$ span C *is a Chebyshev subspace of* $\mathscr{C}(\mathscr{H})$ *if and only if* $0 \notin \pi_0(C) \cap \pi_0(C^*)$.

Proof. Suppose V is not Chebyshev. By Theorem 1, there exists $0 \neq C \in V$, which vanishes on a generalized α set. Hence, $0 \in \pi_0(C) \cap \pi_0(C^*)$.

Now suppose $0 \in \pi_0(C) \cap \pi_0(C^*)$. Select unit vectors u, v such that $C(v) = 0 = C^*(u)$. Define $T = v \otimes \bar{u}$. Since $T(C) = \langle Cv, u\rangle = 0$, $T \in S(V^\perp)$ and C is zero on the generalized θ set associated with T, then V is not Chebyshev by Theorem 2. Q.E.D.

We now give an intrinsic characterization of the finite-dimensional Chebyshev subspaces of $\mathscr{C}(\mathscr{H})$.

THEOREM 5. *An N-dimensional subspace* $V \subset \mathscr{C}(\mathscr{H})$ *is Chebyshev if and only if there does not exist a nonzero* $C \in V$, $C_j \in V$, $j = 1,\ldots, N-1$, *and two sets A and B each consisting of m orthonormal elements so that*

(i) $\operatorname{span}(C, C_1, \ldots, C_{N-1}) = V$

(ii) $O \neq A = \{v_1, \ldots, v_m\} \subset \ker C$, $\quad B = \{u_1, \ldots, u_m\} \subset \ker C^*$

(iii) *the* $(N-1) \times m$ *matrix*

$$M = (\langle C_i v_j, u_j\rangle)_{i=1,\ldots,N-1; j=1,\ldots,m}$$

has linearly dependent columns.

Proof. Suppose V is not Chebyshev. By Theorem 2, there exists a nonzero $C \in V$ that vanishes on the generalized θ set associated with some $T \in S(V^\perp)$, where $T = \sum_1^M \lambda_n v_n \otimes \bar{u}_n$ and m is finite since T attains its trace norm on $S(\mathscr{C}(\mathscr{H}))$. Pick $C_1, \ldots, C_{N-1}$ so that $\operatorname{span}\{C, C_1, \ldots, C_{N-1}\} = V$. Let $A = \{v_1, \ldots, v_m\}$ and $B = \{u_1, \ldots, u_m\}$. Since C vanishes on the generalized θ set associated with T, $A \subset \ker C$ and $B \subset \ker C^*$. Also, since $T \in S(V^\perp)$,

$$0 = T(C_j) = \sum_{i=1}^{m} \langle TC_j v_i, v_i\rangle = \sum_{i=1}^{m} \langle C_j v_i, T^* v_i\rangle = \sum_{i=1}^{m} \lambda_i \langle C_j v_i, u_i\rangle.$$

Thus, if M_j denotes the jth column of M, we have $\lambda_1 M_1 + \cdots + \lambda_m M_m = 0$.

Conversely, suppose there exists a nonzero $C \in V$ so that $Cv_j = 0 = C^* u_j$, $j = 1,\ldots, m$, and $\lambda_1 M_1 + \cdots + \lambda_m M_m = 0$. Without loss of generality, assume $\sum_{k=1}^m |\lambda_k| = 1$. If $\lambda_k = p_k e^{i\theta_k}$ in its polar decomposition, set $v_k^1 = e^{i\theta_k} v_k$; thus, we have $p_1 M_1 + \cdots + p_m M_m = 0$ with $p_k > 0$ for all k. Now define

$$T = \sum_1^m p_k v_k^1 \otimes \bar{u}_k .$$

It is easy to check that $T \in S(V^{\perp})$ and that C is zero on the associated generalized θ set. By Theorem 2, V is not Chebyshev. Q.E.D.

COROLLARY 1. *Let V be an N-dimensional subspace of $\mathscr{C}(\mathscr{H})$ and suppose there exists a nonzero $C \in V$ so that* dim ker C *and* dim ker $C^* \geqslant N$. *Then V is not Chebyshev.*

COROLLARY 2. *If $\mathscr{H}$ is not separable, $\mathscr{C}(\mathscr{H})$ has no finite-dimensional Chebyshev subspaces.*

Proof. The result follows immediately from the preceding corollary and the Schmidt decomposition of a compact operator.

Many times, an intrinsic characterization of the finite-dimensional Chebyshev subspaces of a space is obtained by showing that these subspaces are interpolating. However, it will be shown that this is not the case in $\mathscr{C}(\mathscr{H})$.

DEFINITION 3. An N-dimensional subspace V of a Banach space E is an interpolating subspace if for every linearly independent set $\{Q_1, ..., Q_N\} \subset$ extreme points of $U(E^*)$ and every set $\{a_1, ..., a_N\}$ of scalars, there is a unique $y \in V$ for which $Q_i(y) = a_i$, $i = 1, ..., N$.

It is well known [1] that the finite-dimensional Chebyshev subspaces of $C(X)$ and c_0 are precisely the interpolating subspaces of those spaces. An analogous situation might be hoped for in $\mathscr{C}(\mathscr{H})$. In [11], Singer characterized the extreme points of the dual unit ball of any tensor product Banach space normed with the inductive limit topology. For completeness, we give a simple proof of the characterization of the extreme points of the unit ball of τ.c. which is a special case of Singer's theorem.

THEOREM 6. *The extreme points of the unit ball of τ.c., denoted by $\mathscr{E}$, are the rank one operators.*

Proof. Suppose $|||\, T \,||| = 1$ and rank $T \geqslant 2$. Then $T = \sum \lambda_n v_n \otimes \bar{u}_n$ with $1 > \lambda_1 \geqslant \lambda_2 > 0$. Pick $\epsilon > 0$ so that $\lambda_1 + \epsilon < 1$ and $\lambda_2 - \epsilon > 0$. Consider

$$T_1 = (\lambda_1 + \epsilon)\, v_1 \otimes \bar{u}_1 + (\lambda_2 - \epsilon)\, v_2 \otimes \bar{u}_2 + \sum_{n \geqslant 3} \lambda_n v_n \otimes \bar{u}_n$$

and

$$T_2 = (\lambda_1 - \epsilon)\, v_1 \otimes \bar{u}_1 + (\lambda_2 + \epsilon)\, v_2 \otimes \bar{u}_2 + \sum_{n \geqslant 3} \lambda_n v_n \otimes \bar{u}_n .$$

Then $|||\, T_1 \,||| = |||\, T_2 \,||| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$ so that $T \notin \mathscr{E}$.

Conversely, let $T = v \otimes \bar{u}$ be a rank one operator and $\| u \| = \| v \| = ||| T ||| = 1$. Suppose there exists T_1 and T_2 such that $||| T_1 ||| = ||| T_2 ||| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Now we have

$$1 = T(u \otimes \bar{v}) = (u \otimes \bar{v})(T) \leqslant \| u \otimes \bar{v} \|| \left(\frac{\| T_1 \|}{2} + \frac{\| T_2 \|}{2} \right) \leqslant 1.$$

Since $\| T_i \| \leqslant ||| T_i ||| = 1, i = 1, 2,$ it follows that $1 = ||| T_1 ||| = \| T_1 \|$. Thus, T is a rank 1 operator by the Schmidt decomposition of T. By Lemma 2, $T_1(u) = v$, hence, $T_1 = v \otimes \bar{u} = T$. Similarly, it follows that $T_2 = T$. Q.E.D.

PROPOSITION 1. *There exists no finite-dimensional interpolating subspace of* $\mathscr{C}(\mathscr{H})$.

Proof. Let V be an N-dimensional subspace of $\mathscr{C}(\mathscr{H})$ and let $0 \neq K \in V$ with Schmidt decomposition $K = \sum_{j=1}^{m} \lambda_j v_j \otimes \bar{u}_j$. Without loss of generality, assume $m \geqslant N + 1$. Let

$$T_k = u_1 \otimes \bar{v}_{k+1}, \qquad k = 1,..., N.$$

Thus,

$$\sum \langle T_k K u_j , u_j \rangle = \sum_j \langle K u_j , T_k^* u_j \rangle = \langle v_1 , v_{k+1} \rangle = 0, \qquad \text{for each } k.$$

Thus, T_k annihilates K for each k; and hence, V is not interpolating. Q.E.D.

Proposition 1 shows that $\mathscr{C}(\mathscr{H})$ is in stark contrast with $C(X)$ and c_0, which both have interpolating subspaces.

It is clear that $\mathscr{C}(\mathscr{H})$ fails to have interpolating subspaces because $\mathscr{E}$ has "too many" elements. Various attempts to find an analogous concept of interpolating subspace in $\mathscr{C}(\mathscr{H})$ proved futile.

3. CHEBYSHEV SUBSPACES OF FINITE CODIMENSION

To characterize the Chebyshev subspaces of finite codimension in $\mathscr{C}(\mathscr{H})$, we first characterize the proximinal subspaces of $\mathscr{C}(\mathscr{H})$. The following theorem is based on two well-known results of Garkavi [10]:

(a) A factor-reflexive linear subspace G of a normed linear space E is proximinal if and only if for each

$$\Phi \in (G^\perp)^*, \qquad \exists\, y \in E$$

such that

$$\Phi(f) = f(y), \qquad \forall f \in G^\perp$$

and

$$\| \Phi \| = \| y \|.$$

(b) If G is proximinal and factor reflexive, $f \in G^{\perp}$ attains its norm on the unit ball of E.

THEOREM 8. *If $V \subset \mathscr{C}(\mathscr{H})$ is a subspace of finite codimension, then V is proximinal in $\mathscr{C}(\mathscr{H})$ if and only if $V^{\perp}$ has a basis of finite-rank operators.*

Proof. If V is proximinal, then by (b) above, every $T \in V^{\perp}$ attains its norm implying every $T \in V^{\perp}$ is of finite rank by Lemma 2.

Conversely, suppose the elements of a basis of $V^{\perp}$ are of finite rank. Let $H \subset \mathscr{H}$ be the smallest subspace of $\mathscr{H}$ containing the ranges of the elements of a basis of $V^{\perp}$ and the ranges of their adjoints, i.e., $H = \operatorname{span}\{\mathscr{R}(T), \mathscr{R}(T^*)\colon T \in V^{\perp}\}$. Note that $\dim H < \infty$. Thus, any $T \in V^{\perp}$ may be written as $T = \hat{T} \otimes 0$ where $\hat{T}$ is an operator matrix in $\tau.\text{c.}(H)$ and $|||\, T \,||| = |||\, \hat{T} \,|||$.

Since $V^{\perp}$ may be identified isometrically with a subspace of $\tau.\text{c.}(H)$, each $S \in (V^{\perp})^*$ may be identified norm preservingly with an $\hat{S} \in (\tau.\text{c.}(H))^*$. By a theorem of Schatten [9], $\mathscr{B}(H) = (\tau.\text{c.}(H))^*$. Thus, given $S \in (V^{\perp})^*$ with corresponding $\hat{S}$ in $\mathscr{B}(H)$ and given any $T \in V^{\perp}$ with corresponding $\hat{T}$ in $\tau.\text{c.}(H)$, we may write

$$S(T) = \operatorname{trace}(\hat{S}\hat{T}) = \operatorname{trace}(\hat{T}\hat{S}) = T(\hat{S} \otimes 0),$$

Since $\hat{S} \otimes 0$ has finite rank, $\hat{S} \otimes 0 \in \mathscr{C}(\mathscr{H})$, and

$$\| S \| = \| \hat{S} \|_{\mathscr{B}(H)} = \| \hat{S} \otimes 0 \|_{\mathscr{C}(\mathscr{H})}. \qquad \text{Q.E.D.}$$

COROLLARY. *$\mathscr{C}(\mathscr{H})$ has no Chebyshev subspace of finite codimension.*

Proof. Without loss of generality, assume V is proximinal in $\mathscr{C}(\mathscr{H})$ and of finite codimension. Let H be as in the proof of Theorem 8, $T \in S(V^{\perp})$, and $u \in H^{\perp}$, $\| u \| = 1$. Define $C = u \otimes \bar{u}$, $C \in S(V)$. Clearly $C = 0$ on the generalized θ set associated with T so V is not Chebyshev by Theorem 2. Q.E.D.

Our last theorem shows to what degree closed subspaces of finite codimension are non-Chebyshev. In what follows, $\mathscr{P}(T)$ will denote the set of best approximants (b.a.'s) to T from V.

THEOREM 9. *Let $V \subset \mathscr{C}(\mathscr{H})$ be any closed subspace of finite codimension and $T \in \mathscr{C}(\mathscr{H}) \backslash V$. Then $\mathscr{P}(T) = \varnothing$ or $\mathscr{P}(T)$ is of infinite dimension.*

Proof. Let K be a b.a. of T from V. We construct an infinite-dimensional set of C_n's $\in V \ni \| T - K \| = \| T - (K + C_n) \|$. Since $T - K$ is a nonzero compact operator $\sigma(| T - K |)$ and $\sigma(|(T - K)^*|)$ contain at least two points (0 and $\| T - K \|$). Let $E(\lambda)$ and $F(\lambda)$ be the corresponding spectral resolutions of $|T - K|$ and $|(T-K)^*|$. Now pick $0 < b < \|T - K\|$. Then $E[0, b]$ and $F[0, b]$ are nonzero spectral projections and it is easy to see that

$$E\mathscr{H} = E[0, b]\,\mathscr{H}, \qquad F\mathscr{H} = F[0, b]\,\mathscr{H}$$

are finite-codimensional subspaces of $\mathscr{H}$. Let

$$\mathscr{M} = \{S \in \mathscr{C}(\mathscr{H}) : S\mathscr{H} \subset F\mathscr{H},\ S(I - E)\,\mathscr{H} = 0\}.$$

It is clear that $\mathscr{M}$ is an infinite-dimensional subspace of $\mathscr{C}(\mathscr{H})$. Since V is of finite codimension pick an infinite-dimensional set $C_n \in V \cap \mathscr{M}$, $C_n \neq 0$, $\|C_n\| \leqslant \epsilon$ where $b + \epsilon \leqslant \|T - K\|$. Now for such C_n, since $F(T - K) = (T - K)E$ and $(I - F)(T - K) = (T - K)(I - E)$, we have

$$\begin{aligned}
&\|T - [K + C_n]\| \\
&\quad = \|F[T - (K + C_n)]\,E + F[T - (K + C_n)](I - E) \\
&\qquad + (I - F)[T - (K + C_n)]\,E + (I - F)[T - (K + C_n)](I - E)\| \\
&\quad = \|F[T - (K + C_n)]\,E + F(I - F)[T - K] + (I - F)\,F[T - K] \\
&\qquad + (I - F)[T - K](I - E)\| \\
&\quad = \max\{\|F[T - K]\,E + FC_nE\|, \|(I - F)[T - K](I - E)\|\} \\
&\quad = \max\{b + \epsilon, \|T - K\|\} = \|T - K\|.
\end{aligned}$$

Q.E.D.

References

1. D. Ault, *et al.*, *J. Approximation Theory* **3** (1970), 164–182.
2. E. W. Cheney and D. E. Wulbert, *Math. Scand.* **24** (1969), 113–140.
3. J. Dixmier, *Ann. Math*, **51** (1950), 387–408.
4. P. R. Halmos, *Indiana Univ. Math. J.* **21** (1972), 951–960.
5. R. Holmes, B. Scranton, and J. Ward, *Bull. Amer. Math. Soc.* **80** (1974), 98–102.
6. C. Olsen, *Indiana Univ. Math. J.* to appear.
7. R. R. Phelps, *Pac. J. Math.* **13** (1963), 647–655.
8. D. Rogers, *Not. Amer. Math. Soc.* **20** (1973), A-429.
9. R. Schatten, "Norm Ideals of Completely Continuous Operators," Springer-Verlag, New York, 1960.
10. I. Singer, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer, Berlin-Heidelberg, 1970.
11. I. Singer, *Bull. Des Sciences Math.* **82** (1958), 73–80.